Leibnitz's Theorem (Some Example)

1) If $y = (\sin^{-1}x)^2$, Prove that

$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - n^2 y_n = 0$

Proof. we have $y = (\sin^{-1}x)^2$

Differentiating both sides w.r.t. x, we get

$$y_1 = 2(\sin^{-1}x)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow y_1\sqrt{1-x^2} = 2\sin^{-1}x$$

$$\Rightarrow y_1^2(1-x^2) = 4(\sin^{-1}x)^2$$

$$\Rightarrow y_1^2(1-x^2) = 4y \quad [\because y = (\sin^{-1}x)^2]$$

Differentiating both sides w.r.t. x, we get

$$2y_1y_2(1-x^2) + y_1^2(-2x) = 4y_1$$

$$\Rightarrow \cancel{2y_1}[y_2(1-x^2) - xy_1] = \cancel{4y_1}\,2$$

$$\Rightarrow y_2(1-x^2) - xy_1 = 2$$

Differentiating both sides n times w.r.t. x and using Leibnitz's theorem, we have

$$[y_{n+2}(1-x^2) + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2)]$$
$$-[y_{n+1}x + {}^nC_1\,y_n(1)] = 0$$

$$\Rightarrow \left[y_{n+2}(1-x^2) + n y_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)\right] - [y_{n+1}x + n y_n] = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - (2n+1)x\,y_{n+1} - [n(n-1)+n]\,y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - n^2 y_n = 0$$; Proved.

2) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, Prove that

$(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0.$

Proof:- Given, $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$

Diff. both sides w.r.t. x, we get

$\frac{1}{m} y^{\frac{1}{m}-1} y_1 + \left(-\frac{1}{m}\right) y^{-\frac{1}{m}-1} y_1 = 2$

$\Rightarrow \frac{1}{m} \frac{y^{\frac{1}{m}}}{y} y_1 - \frac{1}{m} \frac{y^{-\frac{1}{m}}}{y} y_1 = 2$

$\Rightarrow y_1 \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right] = 2my$

$\Rightarrow y_1^2 \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right]^2 = 4m^2y^2$ [Squaring both sides]

$\Rightarrow y_1^2 \left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4y^{\frac{1}{m}} \cdot y^{-\frac{1}{m}}\right] = 4m^2y^2$

$\Rightarrow y_1^2 \left[(2x)^2 - 4\right] = 4m^2y^2$

$\Rightarrow \cancel{4}\, y_1^2 (x^2-1) = \cancel{4}\, m^2y^2 \Rightarrow y_1^2(x^2-1) = m^2y^2$

Diff. both sides w.r.t x, we get

$2y_1y_2(x^2-1) + y_1^2(2x) = m^2(2yy_1)$

$\Rightarrow \cancel{2y_1}\left[y_2(x^2-1) + xy_1\right] = \cancel{2y_1}(m^2y)$

$\Rightarrow y_2(x^2-1) + xy_1 = m^2y.$

Diff. n times on both sides w.r.t. x and by Leibnitz's theorem

$\left[y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2)\right] + \left[y_{n+1}x + {}^nC_1\, y_n(1)\right] = m^2y_n$

$\Rightarrow \left[y_{n+2}(x^2-1) + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2)\right] + \left[y_{n+1}x + ny_n\right] = m^2y_n$

$\Rightarrow y_{n+2}(x^2-1) + (2n+1)\,y_{n+1} + (n^2-m^2)\,y_n = 0$

; proved